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H. BRUNNER

SUPERCONVERGENCE IN COLLOCATION AND IMPLICIT RUNGE-KUTTA METHODS FOR VOLTERRA-TYPE INTEGRAL EQUATIONS OF THE SECOND KIND

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Superconvergence in collocation and implicit Runge-Kutta methods for Volterra-type integral equations of the second kind*)

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H. Brunner

ABSTRACT

Collocation methods in certain piecewise polynomial spaces for Volterra and Abel integral equations of the second kind lead to implicit, semi-implicit, or explicit Runge-Kutta methods of Pouzet-type if the collocation equation is discretized by appropriate numerical quadrature. This paper deals with results on global convergence and (local) superconvergence of such methods; it turns out that there is no complete analogy between these results and those for implicit Runge-Kutta methods for ordinary differential equations, and the local superconvergence property (at the knots) is lost entirely if the kernel of the equation is weakly singular. Finally, we illustrate, by means of an example arising in a modelling process in heart physiology, that a certain superconvergence result for first-kind Volterra equations can be used to deal with oscillatory components in the numerical approximation of the exact solution.

KEY WORDS & PHRASES: *Volterra and Abel integral equations of the second kind, collocation, implicit Runge-Kutta methods, local superconvergence*

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*) This report will be submitted for publication elsewhere.

Collocation methods in certain piecewise polynomial spaces for Volterra integral equations of the second kind yield implicit, semi-implicit, or explicit Runge-Kutta methods of Pouzet type if the collocation equation is discretized by appropriate numerical quadrature. This paper deals with results on global convergence and (local) superconvergence for these methods, both for equations with regular and with weakly singular kernels. In addition, we consider analogous methods for a particular Volterra integral equation of the first kind arising in a modelling process in heart physiology.

I. INTRODUCTION

Let $I := [t_0,T]$ be a compact interval ($t_0<T$), and let Δ_N denote the (uniform) partition of I given by $t_n = t_0 + nh$, $n = 0,1,\dots,N$ ($t_N=T$, $N\geq 1$). Furthermore, set $Z_N := \{t_n : n=1,\dots,N\}$, $\sigma_n := (t_n,t_{n+1}]$ ($n=1,\dots,N-1$), with $\sigma_0 := [t_0,t_1]$, and define the piecewise polynomial spaces $S_m^{(d)}(Z_N)$ with the integer d satisfying $-1\leq d<m$) by

$$(1.1)\qquad S_m^{(d)}(Z_N) := \{u : u \in C^d(I), u|_{\sigma_n} \in \pi_m (n=0,1,\dots,N-1)\}.$$

The special choice $d = -1$ yields the space of piecewise polynomials of degree m which may possess (finite) discontinuities at their knots Z_N. We have $\dim S_m^{(d)}(Z_N) = N(m-d)+(d+1)$.

For a given Volterra integral equation of the second kind,

$$(1.2)\qquad y(t) = g(t) + \int_{t_0}^{t} K(t,\tau,y(\tau))d\tau,\ t \in I,$$

where $K(t,\tau,y)$ is assumed to be continuous on $S \times \mathbb{R}$ ($S := \{(t,\tau) : t_0\leq\tau\leq t\leq T\}$), $g \in C(I)$, we wish to determine an element $u \in S_m^{(d)}(Z_N)$ ($d \in \{-1,0\}$) which satisfies (1.2) on some appropriate finite subset $X(N)$ of I (*collocation* on $X(N)$). This set of collocation points,

$X(N) = \bigcup_{n=0}^{N-1} X_n$, is characterized by

(1.3) $$X_n := \{\xi_{n,j} = t_n + c_j h:\ 0 \leqq c_1 < \ldots < c_{m-d} \leqq 1\}$$

Hence, the resulting collocation equation for u(t) may be written in the form

(1.4) $$u_n(\xi_{n,i}) = h \int_0^{c_i} K(\xi_{n,i}, t_n+sh,\ u_n(t_n+sh))ds + \tilde{F}_n(\xi_{n,i})$$

$$(i=1,\ldots,m-d),$$

with $u_n(t) = u|_{\sigma_n}$, and where

(1.5) $$\tilde{F}_n(t) := g(t) + h \sum_{\ell=0}^{n-1} \int_0^1 K(t, t_\ell+sh,\ u_\ell(t_\ell+sh))ds \qquad (t \geqq t_n)$$

approximates the "tail term" in (1.2), given by

(1.5) $$F_n(t) := g(t) + \int_{t_0}^{t_n} K(t,\tau,y(\tau))d\tau \qquad (t \geqq t_n).$$

If $d > -1$, additional equations are furnished by the continuity conditions for u(t) on Z_N; also, $u_0(t_0) = g(t_0)$. It can be shown (see [3],[19],[12]) that, under the usual smoothness hypotheses on K and y, $u \in S_m^{(d)}(Z_N)$ $(d \in \{-1,0\})$ generated by this collocation scheme satisfies

(1.6) $$\|u-y\|_\infty \leqq \theta_0 h^{m+1} \quad (h\to 0,\ Nh = T-t_0).$$

In the present paper we shall discuss the following questions:

(i) Since the integrals in (1.4) and (1.5) can, in general, not be found analytically, how should one choose the quadrature formulas for their discretization?

(ii) Are there parameters $\{c_i: i = 1,\ldots,m-d\}$ (and corresponding quadrature formulas) such that the *local* rate of convergence, p^*, on Z_N satisfies $p^* > m+1$ (*superconvergence* with respect to the knots Z_N)? If so, does this yield analogous results for (implicit) Runge-Kutta methods for (1.2)?

(iii) How does one have to modify the answer to (i) and (ii) if (1.2) is replaced by

$$(1.7)\qquad y(t) = g(t) + \int_{t_0}^{t} \frac{K(t,\tau,y(\tau))d\tau}{(t-\tau)^{\alpha}}, \quad t \in I,\ 0 < \alpha < 1$$

(*Abel integral equation of the second kind*)?

The paper is organized as follows: in Section 2 we recall certain relevant results on collocation and (implicit) Runge-Kutta methods for ordinary differential equations; there, we also indicate why a certain superconvergence result will not carry over to Volterra integral equations (1.2). Section 3 deals with the connection between collocation methods for (1.2) in $S_m^{(-1)}(Z_N)$ and Runge-Kutta methods of Pouzet type ([28];[4]). In Section 4 we consider the question of superconvergence and investigate the extension of collocation and Runge-Kutta methods to Abel integral equations of the form (1.7). Finally, in Section 5, we apply these methods to a particular Volterra integral equation of the first kind (arising in connection with a modelling problem in heart physiology ([10])): while the above superconvergence results for second-kind Volterra equations do not carry over to equations of the first kind ([8]), it is shown that, in this example, a judicious choice of the collocation points ("quasi-Hermite" collocation) leads to a damping of the initial oscillations (in the numerical solution) which are caused by the given perturbation term in the model equation.

II. PRELIMINARIES

Consider the initial-value problem

$$(2.1)\qquad y' = f(t,y)\ \ (t \in I),\ y(t_0) = y_0,$$

and suppose that its exact solution, $y = y(t)$, is approximated by an element $u \in S_m^{(0)}(Z_N)$ obtained by collocation with respect to $X(N)$ introduced in (1.3)(d=0); i.e. $u = u(t)$ is generated recursively by

$$(2.2a)\qquad u_n(t_n) = u_{n-1}(t_n) \qquad (u_0(t_0) = y_0),$$

$$(2.2b)\qquad u_n'(t_n+c_ih) = f(t_n+c_ih, u_n(t_n+c_ih))$$

$$(i=1,\ldots,m;\ n=0,1,\ldots,N-1).$$

This process represents an m-stage implicit Runge-Kutta method, generating approximations $y_n = u(t_n)$ to $y(t_n)$ $(t_n \in Z_N)$; compare [18],[26],[27]. To see this, let $k_i^{(n)} := u_n'(t_n+c_ih)$ $(i=1,\dots,m)$, and observe that

$$u_n(t_n+c_ih) = y_n + h\int_0^{c_i} u_n'(t_n+\tau h)\,d\tau,$$

where

$$u_n'(t_n+\tau h) = \sum_{j=1}^{m} \ell_j(\tau)u_n'(t_n+c_ih) \qquad (\text{since } u_n' \in \pi_{m-1}),$$

with $\ell_j(\tau) := \prod_{r\neq j}^{m} (\tau-c_r)/(c_j-c_r)$.

Thus, equation (2.2b) may be written as

$$\text{(2.3a)} \qquad k_i^{(n)} = f(t_n+c_ih,\ y_n+h\sum_{j=1}^{m} a_{ij}k_j^{(n)}) \qquad (i=1,\dots,m),$$

while (2.2a) (with n replaced by (n+1)) yields

$$\text{(2.3b)} \qquad y_{n+1} = y_n + h\sum_{j=1}^{m} b_jk_j^{(n)} \qquad (n=0,1,\dots,N-1).$$

Here we have set $a_{ij} := \int_0^{c_i} \ell_j(\tau)d\tau$, $b_j := \int_0^1 \ell_j(\tau)d\tau$.

(On the other hand, it is a well-known fact that the converse of the above statement does not hold in general; compare also [27] for a thorough discussion of this aspect.)

If the m collocation parameters $\{c_i\}$ in (1.3) satisfy $0 \leqq c_1 < \dots < c_m \leqq 1$ but are otherwise arbitrary then $\|u-y\|_\infty = \mathcal{O}(h^r)$ and $\max\{|u(t_n) - y(t_n)|:t_n \in Z_N\} = \mathcal{O}(h^p)$ (as $h\to 0$, $Nh=T-t_0$), with $r=p=m$; i.e. the global order of convergence coincides with the (local) convergence order at the knots Z_N. However, if the $\{c_i\}$ are taken as the zeros of $P_m(2s-1)$ (*Gauss points* for (0,1)), then $\|u-y\|_\infty = \mathcal{O}(h^r)$ and $\max\{|u(t_n)-y(t_n)|:t_n \in Z_N\} = \mathcal{O}(h^{p^*})$, where now $p^* = 2r = 2m$: we obtain *superconvergence of order 2m* at the knots. Compare (also for analogous results: $p^* = 2m-1$ (*Radau* points) and $p^* = 2m-2$ (*Lobatto* points), respectively) [14],[15],[2],[3],[16],[17],[18],[26]; see [20],[21], [25],[23] for related results.

Consider now the integrated form of (2.1), namely

$$y(t) = y_0 + \int_{t_0}^{t} f(\tau, y(\tau))\,d\tau\,, \qquad t \in I; \tag{2.4}$$

this integral equation is a special case of (1.2) (with $\partial K/\partial t \equiv 0$). If (2.4) is solved numerically by collocation in $S_{m-1}^{(-1)}(Z_N)$ $(m \geqq 1)$, using the same set of collocation points as in (2.2) (recall that $\dim S_m^{(0)}(Z_N) = \dim S_{m-1}^{(-1)}(Z_N) + 1$; if $d = -1$, no starting value is needed), then the resulting approximating element $\hat{u}$ to the solution of (2.4) has the following properties (compare also [3],[19],[12],[11]):

(i) $\sup\{|\hat{u}(t)-y(t)| : t \in I\} = \mathcal{O}(h^m)$
for any choice $0 < c_1 < \ldots < c_m \leqq 1$;

(ii) if these $\{c_i\}$ are the *Radau abscissas* (with $c_m=1$) then
$\max\{|\hat{u}(t_n)-y(t_n)| : t_n \in Z_N\} = \mathcal{O}(h^{2m-1})$;

(iii) if the *Gauss points* are chosen as the $\{c_i\}$ (note that here $c_m<1$) then
$\max\{|\hat{u}(t_n)-y(t_n)| : t_n \in Z_N\} = \mathcal{O}(h^m)$
(i.e. collocation for (2.4) based on the Gauss points does, in contrast to the direct use of (2.1), no longer furnish superconvergence at the knots Z_N);

(iv) if $f(t,y) = \lambda y$ (λ = const.) in (2.1) and (2.4), and if $u \in S_m^{(0)}(Z_N)$ and $\hat{u} \in S_{m-1}^{(-1)}(Z_N)$ denote, respectively, the corresponding collocation approximations based on $0 \leqq c_1 \leqq \ldots < c_m = 1$, then $u(t_n) = \hat{u}(t_n)$ for all $t_n \in Z_N$; this statement is no longer valid if $c_m < 1$.

The above results suggest that there is no complete analogy between superconvergence results for the initial-value problem (2.1) and the Volterra integral equation (1.2). The reason for this will be studied in Section 4: since the collocation equation (2.2) (or (1.4)) may be interpreted as a perturbation of the original equation (2.1) (or (1.2)), the *variation of constants formula* associated with the given functional equation can be used to write down the resulting error function; it turns out that there is a crucial difference in the formulas for (2.1) and (1.2) (compare also [1],[31],[27];[24],[7],[6]) which is reflected in the superconvergence results.

III. COLLOCATION AND RUNGE-KUTTA METHODS

An m-stage (implicit) Runge-Kutta method for the Volterra integral equation (1.2) is given by

$$(3.1)\quad \begin{aligned} Y_i^{(n)} &= h\sum_{j=1}^{m} a_{ij}\, K(t_n+c_{ij}h,\ t_n+d_jh,\ Y_j^{(n)}) + \hat{F}_n(t_n+\theta_ih) \quad (i=1,\dots,m+1),\\ y_{n+1} &= Y_{m+1}^{(n)} \qquad (n=0,1,\dots,N-1). \end{aligned}$$

Here, y_{n+1} approximates $y(t_{n+1})$, and $\hat{F}_n(t_n+\theta_ih)$ denotes a suitable approximation (usually generated by a quadrature rule of appropriate degree of precision) to the exact tail term $F_n(t_n+\theta_ih)$ (see (1.5')).

In analogy with Runge-Kutta methods for ODEs we shall often set $b_j: = a_{m+1,j}$, and we assume that $d_i = \Sigma_{j=1}^m a_{ij}$ $(i=1,\dots,m)$.

The above scheme (3.1) contains two important *special cases*.

(A) If $c_{ij} = d_i$ $(j=1,\dots,m;\ i=1,\dots,m+1)$, $\theta_i = d_i$, with $d_1 = 0$ and $d_{m+1} = 1$, we obtain

$$(3.2)\quad \begin{aligned} Y_i^{(n)} &= h\sum_{j=1}^{m} a_{ij} K(t_n+d_ih,\ t_n+d_jh,\ Y_j^{(n)}) + \hat{F}_n(t_n+d_ih) \quad (i=1,\dots,m+1)\\ y_{n+1} &= Y_{m+1}^{(n)} \qquad (n=0,1,\dots,N-1). \end{aligned}$$

This is the implicit version of the Runge-Kutta scheme introduced by Pouzet [28]; if the upper limit of summation in (3.2) is replaced by (i-1) we have the original explicit method of Pouzet where the approximations $\hat{F}_n(t_n+d_ih)$ are generated by a suitable quadrature rule (e.g. Gregory's rule). We also observe that the number of kernel evaluations (per step) in the "Runge-Kutta part" of (3.2) equals $m(m+1)$ (implicit scheme), and $m(m+1)/2$ (explicit scheme), respectively.

(B) If $c_{ij} = c_j$ $(j=1,\dots,m;\ i=1,\dots,m+1)$, with $c_i \geqq d_i$, and $\theta_i = c_i$, then (3.1) becomes

$$(3.3)\quad \begin{aligned} Y_i^{(n)} &= h\sum_{j=1}^{m} a_{ij}\, K(t_n+c_jh,\ t_n+d_jh,\ Y_j^{(n)}) + \hat{F}_n(t_n+c_ih) \quad (i=1,\dots,m+1),\\ y_{n+1} &= Y_{m+1}^{(n)} \qquad (n=0,1,\dots,N-1). \end{aligned}$$

The explicit version of this scheme was introduced by Bel'tyukov [5].

The number of kernel evaluations (per step) in the Runge-Kutta part of (3.3) is equal to m, independent of whether the method is implicit or explicit.

As an illustration we give two simple examples corresponding to m = 1 (compare also [13] for a more detailed list of Runge-Kutta methods).

(a) Midpoint method of *Pouzet* type: here, we have

$$(3.4)\qquad \begin{aligned} Y_1^{(n)} &= \frac{h}{2} K(t_n + \frac{h}{2}, t_n + \frac{h}{2}, Y_1^{(n)}) + \hat{F}_n(t_n + \frac{h}{2}) \\ y_{n+1} = Y_2^{(n)} &= hK(t_n+h, t_n + \frac{h}{2}, Y_1^{(n)}) + \hat{F}_n(t_n+h), \end{aligned}$$

where $\hat{F}_n(t_n + \frac{h}{2})$ and $\hat{F}_n(t_n+h)$ correspond to the application of the (composite) trapezoidal rule to (1.5'). For each value of n, (3.4) requires two kernel evaluations in the Runge-Kutta part plus two evaluations of the backward term.

(b) Midpoint method of *Bel'tyukov* type:

$$(3.5)\qquad \begin{aligned} Y_1^{(n)} &= \frac{h}{2} K(t_n+h, t_n + \frac{h}{2}, Y_1^{(n)}) + \hat{F}_n(t_n+h), \\ y_{n+1} = Y_2^{(n)} &= hK(t_n+h, t_n + \frac{h}{2}, Y_1^{(n)}) + \hat{F}_n(t_n+h); \end{aligned}$$

here, we have one kernel evaluation plus one evaluation of the tail term. Both methods possess the order p = 2.

If the renewal equation,

$$(3.6)\qquad y(t) = \frac{1}{2} t^2 e^{-2} + \frac{1}{2} \int_0^t (t-s)^2 e^{s-t} y(s)ds, \quad t \in [0,10],$$

whose exact solution is given by

$$y(t) = \frac{1}{3} \{1-e^{-3t/2} (\cos(\frac{t.\sqrt{3}}{2}) + \sqrt{3} . \sin(\frac{t.\sqrt{3}}{2}))\},$$

is solved numerically by (3.4) and (3.5), we obtain the following results given in Table I; due to the special structure of the kernel the errors for the two schemes tend to the same value as $h \to 0_+$.

TABLE I

t_n	$y(t_n)$	$e(t_n) = y_n - y(t_n)$ for: (3.4) (Pouzet)	(3.5) (Bel'tyukov
0.5	.07597	+1.38E-04 -4.71E-06 -8.68E-07	+1.40E-04 -4.71E-06 -8.68E-07
⋮			
2.0	.30763	-1.68E-03 -3.54E-05 -4.90E-06	-1.66E-03 -3.54E-05 -4.90E-06
⋮			
5.0	.33370	-4.06E-03 -7.01E-05 -9.56E-06	-4.01E-03 -7.01E-05 -9.56E-06
⋮			
10.0	.33333	-7.88E-03 -1.30E-04 -1.76E-05	-7.79E-03 -1.30E-04 -1.76E-05

(*Stepsizes:* h = .5, h = .1, h = .05.)

Let us now return to the collocation equation (1.4) associated with the Volterra equation (1.2), and assume that d = -1. We discuss some of the possible discretizations of the integral expressions on the right-hand side of (1.4) (where the information concerning the order of these approximation schemes may in turn be used to choose an appropriate quadrature formula in (1.5)).

(i) *Fully implicit discretization:*

$$h\int_0^{c_i} K(\xi_{n,i},\ t_n+sh,\ u_n(t_n+sh))ds \to h\sum_{j=1}^{m+1} \hat{a}_{ij}K(t_n+c_ih, t_n+c_jh, Y_j^{(n)}) \qquad (i=1,\ldots,m+1) \tag{3.7}$$

with $Y_j^{(n)} := u_n(t_n+c_jh))$. If this quadrature is interpolatory and based on the (m+1) abscissas $\{c_i\}$ we have scheme (3.2) of [19].

(ii) If we use the *m-point discretization*

$$h \int_0^{c_i} K(\xi_{n,i}, t_n+sh, u_n(t_n+sh))ds \to \tag{3.8}$$

$$h \sum_{j=1}^{m} a_{ij} K(t_n+c_ih, t_n+c_jh, Y_j^{(n)})$$

$(i=1,\ldots,m+1)$

(that is, we use only the first m of the $\{c_i\}$ as abscissas in the quadrature formula), then we obtain an m-stage implicit Runge-Kutta method of Pouzet type (see (3.2)). For m=1, $c_1=\frac{1}{2}$, $c_2=1$, we have the midpoint method (3.4).

(iii) If, for fixed c_i, the quadrature formula is based only on the abscissas $\{c_1,\ldots,c_{i-1}\}$ $(i=1,\ldots,m+1)$, we find an explicit Runge-Kutta method of the form originally introduced by Pouzet in [28].

Other possible discretizations include those leading to semi-implicit Pouzet type methods, and the fully implicit ones which use only values of K contained in the region $S \times \mathbb{R}$ on which K is defined.

However, it is obvious from (1.4) and (3.3) that Runge-Kutta methods of Bel'tyukov type (3.3) cannot be obtained by some discretization (based on quadrature) of the collocation equation (1.4) (unless $\partial K/\partial t \equiv 0$: in this case, (1.2) corresponds to an initial-value problem for an ordinary differential equation).

IV. THE QUESTION OF SUPERCONVERGENCE

The collocation equations (2.2) and (1.4) for the initial-value problem (2.1) and the Volterra integral equation (1.2), respectively, may be rewritten (using a slightly different notation) as

$$u'(t) = f(t,u(t)) + \delta(t),\ t \in I\ (u \in S_m^{(0)}(Z_N):\ u(t_0) = y_0), \tag{4.1}$$

and

$$u(t) = g(t) + \int_{t_0}^{t} K(t,\tau,\hat{u}(\tau))d\tau + \hat{\delta}(t),\ t \in I\ (\hat{u} \in S_{m-1}^{(-1)}(Z_N)), \tag{4.2}$$

where $\delta(t) = \delta(t,u(t))$ and $\hat{\delta}(t) = \hat{\delta}(t,\hat{u}(t))$ satisfy

$$(4.3) \qquad \delta(t) = \hat{\delta}(t) = 0 \text{ for } t \in X(N),$$

with the set $X(N)$ defined in (1.3) ($d=0$): $X(N) = \{t_n+c_ih:$ $0 \leqq c_1 < \ldots < c_m \leqq 1\}$. Thus, the error functions $e(t):= u(t)-y(t)$ and $\hat{e}(t):= \hat{u}(t)-\hat{y}(t)$ (with $\hat{y}(t)$ denoting the exact solution of (1.2)) can be represented by applying the appropriate *variation of constants formula* to the original equations (2.2), (1.2) and their *perturbed* forms (4.1), (4.2), respectively. Variation of constants formulas for (nonlinear) ordinary differential equations are given in [1],[30],[31] (see also [18],[27]), while analogous formulas for Volterra integral equations may be found in [24],[7],[30],[6] (compare also [12] and [11]).

The resulting representations for $e(t)$ and $\hat{e}(t)$ then are

$$(4.4) \qquad e(t) = \int_{t_0}^{t} R(t,\tau;\ u(\tau)).\delta(\tau)d\tau, \qquad t \in I;$$

and

$$(4.5) \qquad \hat{e}(t) = \hat{\delta}(t) + \int_{t_0}^{t} \hat{R}(t,\tau;\ \hat{u}(\tau)).\hat{\delta}(\tau)d\tau, \qquad t \in I;$$

with the resolvent kernels R and $\hat{R}$ being determined by the corresponding variational problems (consult the above references for details, including (piecewise) smoothness properties of $\delta,\hat{\delta},R$, and $\hat{R}$).

Let now $t = t_n \in Z_N$ in (4.4) and (4.5), and write

$$(4.6) \qquad e(t_n) = h \sum_{\ell=0}^{n-1} \int_0^1 R(t_n,\ t_\ell+sh;\ u_\ell(t_\ell+sh)).\ \delta(t_\ell+sh)ds$$

and

$$(4.7) \qquad \hat{e}(t_n) = \hat{\delta}(t_n) + h \sum_{\ell=0}^{n-1} \int_0^1 \hat{R}(t_n,t_\ell+sh;u_\ell(t_\ell+sh)).\ \hat{\delta}(t_\ell+sh)ds$$

$$(n=1,\ldots,N).$$

Suppose that the integrals occurring in (4.6) and (4.7) are evaluated by (interpolatory) quadrature based on the *abscissas* $\{t_\ell+c_jh:\ j=1,\ldots,m\}$ (i.e. the *collocation points* in $t \leqq t_n$). If $E_\ell^{(n)}$ and $\hat{E}_\ell^{(n)}$ denote the respective quadrature errors we find (recalling the collocation condition (4.3)!)

$$e(t_n) = h \sum_{\ell=0}^{n-1} E_\ell^{(n)} \qquad (n=1,\dots,N), \tag{4.8}$$

and

$$\hat{e}(t_n) = \hat{\delta}(t_n) + h \sum_{\ell=0}^{n-1} \hat{E}_\ell^{(n)} \qquad (n=1,\dots,N). \tag{4.9}$$

We have thus shown that, for the *initial-value problem* (2.1), the error resulting from collocation in $S_m^{(0)}(Z_N)$ has the same order as the given quadrature formula (see also [26],[27]), while for the *Volterra integral equation* (1.2) this holds only if $c_m=1$ (i.e. $\hat{\delta}(t_n) = 0$ for $t_n \in Z_N$). Therefore, the use of the *Gauss points* (which satisfy $c_1 > 0$ and $c_m < 1$) in (4.8) yields $|\hat{e}(t_n)| = \mathcal{O}(h^m)$ (since $\hat{\delta}(t_n) \neq 0$); these results also confirm the result of Butcher [14] on the superconvergence of order 2m for the m-stage implicit Runge-Kutta-Gauss method, and the corresponding negative result for Volterra equations of the second kind mentioned at the end of Section II.

On the other hand, the use of the m *Radau* abscissas, with $0 < c_1 < \dots < c_m = 1$, implies $|e(t_n)| = \mathcal{O}(h^{2m-1})$ and $|\hat{e}(t_n)| = \mathcal{O}(h^{2m-1})$, respectively.

Furthermore, if in (4.7) we choose the points $\{c_i : i=1,\dots,m-1\}$ as the zeros of $P_{m-1}(2s-1)$ ((m-1) Gauss points), with $c_m = 1$, and if quadrature is based only on the first (m-1) abscissas, then $|\hat{e}(t_n)| = \mathcal{O}(h^{2m-2})$: since this discretization of the collocation equation (1.4) (with m replaced by (m-1)) yields an implicit (m-1)-stage Runge-Kutta method of Pouzet type (compare (3.8)), we obtain the general result that *implicit m-stage Runge-Kutta-Pouzet-Gauss methods have the order $p^*=2m$*. (Observe, however, that a *fully implicit* discretization (3.7) of (1.4) and (4.7), using the *(m+1) Radau points* (i.e. the zeros of $(s-1)P_m^{(1,0)}(2s-1)$), furnishes an approximation of still higher order on Z_N, namely $|e(t_n)| = \mathcal{O}(h^{2m+1})$.)

We conclude this section by a short discussion on the extension of collocation and Runge-Kutta methods to *Abel integral equations of the second kind*, (1.7); for simplicity, assume that the equation is linear:

$$(4.10)\qquad y(t) = g(t) + \int_{t_0}^{t} \frac{G(t,\tau)}{(t-\tau)^{\alpha}}\, y(\tau)d\tau \qquad (0 < \alpha < 1).$$

If (4.10) is solved by collocation in $S_m^{(-1)}(Z_N)$, we obtain

$$(4.11)\qquad u(t) = g(t) + \int_{t_0}^{t} \frac{G(t,\tau)}{(t-\tau)^{\alpha}}\, u(\tau)d\tau + \delta(t), \qquad t \in I,$$

with the defect $\delta(t) = \delta(t,u(t))$ vanishing for $t \in X(N)$ (given by (1.3), $d = -1$). Hence, by a classical result on the resolvent kernel for (4.10), we have (setting $t = t_n \in Z_N$),

$$(4.12)\qquad e(t_n) = \delta(t_n) + h^{1-\alpha} \sum_{\ell=0}^{n-1} \int_0^1 \frac{Q(t_n,t_\ell+sh)}{(n-\ell-s)^{\alpha}} \cdot \delta(t_\ell+sh)ds \qquad (n=1,\dots,N).$$

Note that each integral term contains a weight function depending on the given subinterval σ_ℓ. This fact implies that, in contrast to Volterra equation with regular kernels, we are faced with a *loss of superconvergence* if an Abel equation of the second kind is solved by collocation in $S_m^{(-1)}(Z_N)$ (compare also [12]).

If (4.11) is discretized in analogy to (3.8) (weighted interpolatory quadrature based on (m+1) abscissas $\{t_n+c_ih: i=1,\dots,m+1\}$) we find the following generalization of Pouzet's implicit Runge-Kutta method for (1.7):

$$(4.13)\qquad \begin{aligned} Y_i^{(n)} &= h^{1-\alpha} \sum_{j=1}^{m} a_{ij}(\alpha).K(t_n+c_ih,\ t_n+c_jh, Y_j^{(n)}) + \hat{F}_n(t_n+c_ih) \quad (i=1,\dots,m+1) \\ y_{n+1} &= Y_{m+1}^{(n)} \qquad (n=0,1,\dots,N-1); \end{aligned}$$

$\hat{F}_n(t_n+c_ih)$ is a suitable approximation to

$$(4.14)\qquad F_n(t_n+c_ih) := g(t) + \int_{t_0}^{t_n} \frac{K(t_n+c_ih,\tau,y(\tau))}{(t_n+c_ih-\tau)^{\alpha}} d\tau \ ;$$

note that for $c_1 > 0$, the integrands in (4.14) are no longer weakly singular.

Order conditions for Runge-Kutta methods of the form (4.13) will be derived in an extension of the work reported in [13]; the problem of whether there exist implicit m-stage Runge-Kutta methods for (1.7) with order $p^* \geqq m+2$ is still open. In addition, the *stability properties* of all these (collocation and Runge-Kutta) methods have to be investigated; the recent studies of Wolkenfelt [33] and of van der Houwen and te Riele [32] will be relevant.

V. VOLTERRA EQUATIONS OF THE FIRST KIND: AN EXAMPLE

The results of the previous section indicate that one is faced with a loss of superconvergence (with respect to the knots Z_N) if the given Volterra integral equation of the second kind possesses a weakly singular kernel. A similar result holds for Volterra equations of the first kind, even when the kernel is smooth. Consider

$$(5.1) \qquad \int_{t_0}^{t} K(t,s)y(s)ds = g(t), \quad t \in I;$$

assume that K and g are sufficiently smooth on their respective domains and are such that (6.1) possesses a unique solution $y \in C^r(I)$ for some $r \geqq 0$. If (6.1) is solved numerically by collocation in $S_m^{(-1)}(Z_N)$, using the collocation points X(N) (i.e. (1.3) with d = -1), then ([8],[11]) $\sup\{|u(t)-y(t)|: t \in I\} = \mathcal{O}(h^{m+1})$, provided the collocation parameters $\{c_i\}$ satisfy $\Pi_{j=1}^{m+1}(1-c_j)/c_j < 1$. In addition, it holds that, for any choice of the $\{c_i\}$ which are pairwise distinct and subject to the condition just stated, $\max\{|u(t_n)-y(t_n)|: t_n \in Z_N\} = \mathcal{O}(h^{m+1})$, where (m+1) cannot be replaced by (m+2) (compare [8] for details). Superconvergence (on Z_N) of order $p^* = m+2$ can be attained only if (at least) two collocation parameters coalesce. To see this, consider the integral equation satisfied by the error $e(t) := u(t)-y(t)$; it is

$$(5.2) \qquad \int_{t_0}^{t} K(t,s)e(s)ds = \delta(t), \; t \in I,$$

where $\delta(t) = 0$ for $t \in X(N)$. Under the above assumptions, (5.2) is equivalent to

$$(5.3) \qquad e(t) = [K(t,t)]^{-1}. \{\delta'(t) - \int_{t_0}^{t} K_t(t,s)e(s)ds\},$$

and its solution is thus given by

$$(5.4)\qquad e(t) = [K(t,t)]^{-1}.\delta'(t) + \int_{t_0}^{t} R(t,s)\delta'(s)ds$$
$$= [K(t,t)]^{-1}.\delta'(t) + R(t,t)\delta(t) - \int_{t_0}^{t} R_s(t,s)\delta(s)ds;$$

here, we have used integration by parts and the fact that the defect $\delta(t)$ satisfies $\delta(t_0) = 0$; $R(t,s)$ denotes the resolvent kernel for $K_t(t,s)/K(t,t)$. Setting $t = t_n \in Z_N$ in (5.4), and using arguments similar to those of Section 4, we see that $\max\{|e(t)|: t \in Z_N\} = \mathcal{O}(h^{m+2})$ implies $\delta(t_n) = \delta'(t_n) = 0$ for $t_n \in Z_N$ and hence $c_m = c_{m+1} = 1$.

It is clear that Hermite type collocation for solving (6.1) is of little practical value since it requires knowledge of the differentiated form of (6.1): if this can be found, then (according to the results of Section 4) it will be advantageous to use ordinary collocation to solve this resulting second-kind Volterra equation.

However, it turns out that what might be called "quasi-Hermite" collocation (where two or more of the m collocation parameters $\{c_1,\ldots,c_m\}$ are chosen near $c_{m+1} = 1$) yields methods for solving "difficult" Volterra integral equations of the first kind for which classical methods (e.g. finite-difference methods requiring starting values, or the block methods of Keech [22]) fail. We illustrate this by an example arising in a modelling process in heart physiology (compare [29],[10]). The (slightly modified) equation is

$$(5.5)\qquad \int_0^t F(t-s)y(s)ds = g(t), \qquad t \in [0,1],$$

where

$$F(u) := \begin{cases} \sin(2\pi u), & 0 \leqq u \leqq \tfrac{1}{2}, \\ 0, & \tfrac{1}{2} < u < 1, \end{cases}$$

and

$$g(t) := \begin{cases} \varepsilon.\sin(2\pi t) + g_1(t), & 0 \leqq t \leqq \tfrac{1}{2}, \\ g_2(t), & \tfrac{1}{2} < t \leqq 1;\ 0 \leqq \varepsilon \leqq 1, \end{cases}$$

with

$$g_1(t) := \beta.(e^{-t} + \sin(2\pi t)/(2\pi) - \cos(2\pi t)),$$

$$g_2(t) := \beta.(1+\sqrt{e}).e^{-t}, \qquad \beta := 2\pi/(1+4\pi^2).$$

For $\varepsilon=0$, we have $y(t) = e^{-t}$, while for $\varepsilon=1$ (corresponding to the original model equation), $y(t) = \delta_0(t) + e^{-t}$, with $\delta_0(t)$ denoting the Dirac δ-function.

If $\varepsilon > 0$, the term $\varepsilon.\sin(2\pi t)$ gives rise to severe oscillations in the numerical solution. In order to illustrate this, together with the damping effect occurring in quasi-Hermite collocation, we solved (5.5) in $S_2^{(-1)}(Z_N)$; the resulting moment integrals were evaluated analytically. Table II contains a selection of numerical results.

TABLE II

t	y(t)	e(t) = u(t) - y(t) for: (A) $\varepsilon = 0$	(A) $\varepsilon = 10^{-3}$	(A) $\varepsilon = 1$	(B) $\varepsilon = 0$	(B) $\varepsilon = 10^{-3}$	(B) $\varepsilon = 1$
0.00	-	-4.56E-07	2.20E-01	2.20E+02	-1.97E-06	1.21E-01	1.21E+02
		(-3.59E-00	(1.10E-01	(1.10E+02)	(-1.54E-05	(6.02E-02)	(6.02E+01)
0.05_-	.95123	2.93E-06	2.20E-01	2.20E+02	6.17E-08	1.25E-03	1.25E+00
		(-2.22E-06)	(-2.50E-02)	(-2.50E+01	(3.62E-06)	(-4.96E-04)	(-5.00E-01)
0.05_+	.95123	-2.92E-06	-2.20E-01	-2.20E+02	-1.99E-06	-2.69E-05	-2.50E-02
		(-2.22E-06	(-2.50E-02)	(-2.50E+01)	(3.62E-06)	(-4.96E-04)	(-5.00E-01)
0.10_-	.90484	3.05E-07	-2.20E-01	-2.20E+02	5.79E-08	-2.00E-07	-2.57E-04
		(2.28E-05)	(1.10E-01)	(1.10E+02)	(4.76E-07)	(6.27E-04)	(6.27E-01)
0.10_+	.90484	-2.91E-07	2.20E-01	2.20E+02	-1.79E-06	-1.78E-06	4.82E-06
⋮	⋮	(-2.26E-05)	(-1.10E-01)	(-1.10E+02)	(-1.40E-05)	(-2.65E-05)	(1.25E-02)
		⋮	⋮	⋮	⋮	⋮	⋮
0.20_-	.81873	1.55E-07	-2.20E-01	-2.20E+02	5.28E-08	5.91E-08	6.48E-06
		(1.30E-06)	(-1.10E-01)	(-1.10E+02)	(4.31E-07)	(3.01E-07)	(-1.30E-04)
0.20_+	.81873	-1.42E-07	2.20E-01	2.20E+02	-1.61E-06	-1.62E-06	-7.94E-06
⋮	⋮	(-1.10E-06)	(1.10E-01)	(1.10E+02)	(-1.26E-05)	(-1.26E-05)	(-1.02E-05)
		⋮	⋮	⋮	⋮	⋮	⋮
0.50_-	.60653	-2.15E-07	-2.20E-01	-2.20E+02	3.73E-08	3.98E-08	8.34E-08
		(1.86E-05)	(1.10E-01)	(1.10E+02)	(3.19E-07)	(3.19E-07)	(2.12E-07)
0.50_+	.60653	2.24E-07	2.20E-01	-2.20E+02	-1.20E-06	-1.20E-06	-2.12E-06
⋮	⋮	(-1.85E-05)	(-1.10E-01)	(-1.10E+02)	(-9.37E-06)	(-9.37E-06)	(-8.98E-06)
		⋮	⋮	⋮	⋮	⋮	⋮
1.00	.36788	-6.31E-07	-2.20E-01	-2.20E+02	2.95E-08	2.35E-08	-1.72E-08
		(-5.01E-06)	(-1.10E-01)	(-1.10E+02)	(1.94E-07)	(1.94E-07)	(3.01E-07)

Collocation parameters: (A) $\{c_i\} = \{\frac{1}{3}, \frac{2}{3}, 1.\}$ *Stepsizes:* h = 0.05 (h=0.1).

(B) $\{c_i\} = \{.98, .99, 1.\}$.

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